# ON MINIMAL LINES AND CONGRUENCES IN FOUR-DIMENSIONAL SPACE*

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A study of the $\infty^5$ minimal lines in four-dimensional space and the configurations obtained by considering certain ensembles of these will yield interesting results, if we employ a method similar to that used by LIE † in the case of minimal lines of ordinary space.

## § 1. *The minimal cone.*

Consider a minimal element-cone in space $S_4$; its equation may be written

$$(1) \qquad dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 = 0.$$

A flat space

$$(2) \qquad AX_1 + BX_2 + CX_3 + DX_4 + E = 0$$

is said to be tangent to (1) whenever the following relation between the coefficients holds:

$$(3) \qquad A^2 + B^2 + C^2 + D^2 = 0.$$

We shall call (2) a minimal space, when (3) is satisfied. There exist therefore $\infty^3$ minimal spaces.

We shall now express $A$, $B$, $C$, $D$ as functions of two parameters so that (3) is identically satisfied, by writing

$$A = s + t, \qquad B = i(t - s), \qquad C = st - 1, \qquad D = i(st + 1),$$

so that the equation of the minimal space takes the form

$$(4) \qquad (s + t)X_1 + i(t - s)X_2 + (st - 1)X_3 + i(st + 1)X_4 + E = 0.$$

If now we wish to obtain a minimal $M_3$, we put $E = F(s, t)$ in (4) and differentiate with respect to $s$ and $t$. We have then the following system

$$(5) \qquad \begin{aligned} (s + t)X_1 + i(t - s)X_2 + (st - 1)X_3 + i(st + 1)X_4 + F &= 0, \\ X_1 - iX_2 + tX_3 + itX_4 + F'_s &= 0, \\ X_1 + iX_2 + sX_3 + isX_4 + F'_t &= 0, \end{aligned}$$

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† LIE-SCHEFFERS, *Geometrie der Berührungstransformationen*, vol. 1, chap. 10, pp. 411–480, and chap. 12.

from which by eliminating $s$ and $t$ we get the required minimal manifoldness $M_3$ expressed in cartesian coördinates. If $s$ and $t$ are considered fixed, we have the equations of a minimal line, which may be written, solving (5) for $X_1$, $X_2$, and $X_3$,

$$\begin{aligned} X_1 &= \frac{-i(s+t)}{1+st} X_4 - \frac{(s+t)F + (1-s^2)F'_s + (1-t^2)F'_t}{2(1+st)}, \\ X_2 &= \frac{t-s}{1+st} X_4 + i\frac{(s-t)F - (1+s^2)F'_s + (1+t^2)F'_t}{2(1+st)}, \qquad (6) \\ X_3 &= \frac{-i(st-1)}{1+st} X_4 + \frac{F - sF'_s - tF'_t}{1+st}. \end{aligned}$$

If we consider $s$, $t$, $F$, $F'_s$, $F'_t$ as the coördinates of the minimal line (6), we see at once that *in four-dimensional space there exist* $\infty^5$ *minimal lines, and through any fixed point pass* $\infty^2$ *such lines.*

If

$$F = ast + bs + ct + d,$$

we obtain the following system of lines

$$\begin{aligned} X_1 &= \frac{-i(s+t)}{1+st} X_4 - \frac{(b+c)(1+st) + (a+d)(s+t)}{2(1+st)}, \\ X_2 &= \frac{t-s}{1+st} X_4 + i\frac{(c-b)(1+st) + (a+d)(s-t)}{2(1+st)}, \qquad (7) \\ X_3 &= \frac{i(1-st)}{1+st} X_4 + \frac{d-ast}{1+st}, \end{aligned}$$

which may be written in the form

$$\begin{aligned} X_1 + \frac{b+c}{2} &= -\frac{i(s+t)}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right], \\ X_2 - \frac{i(c-b)}{2} &= \frac{t-s}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right], \qquad (8) \\ X_3 - \frac{d-a}{2} &= \frac{i(1-st)}{1+st}\left[X_4 - \frac{i(a+d)}{2}\right]. \end{aligned}$$

Eliminating $s$ and $t$, we obtain the minimal cone

$$\left(X_1 + \frac{b-c}{2}\right)^2 + \left(X_2 - \frac{i(c-b)}{2}\right)^2 + \left(X_3 - \frac{d-a}{2}\right)^2 + \left(X_4 - \frac{i(a+d)}{2}\right)^2 = 0,$$

which may be considered as the ensemble of all the minimal lines passing through the point

$$-\frac{b+c}{2},\ \frac{i(c-b)}{2},\ \frac{d-a}{2},\ \frac{i(d+a)}{2}.$$

If now we consider $F$, $s$ and $t$ as cartesian coördinates of a three-dimensional space $S_3$, we may state the result obtained thus:

*To all the* $\infty^4$ *paraboloids* $F = ast + bs + ct + d$ *in the space* $S_3$ *there correspond* $\infty^4$ *minimal cones in* $S_4$.

*To all the* $\infty^2$ *points of the paraboloid correspond the* $\infty^2$ *minimal lines which pass through the point*

$$(9) \quad \alpha = -\frac{c+b}{2}, \qquad \beta = \frac{i(c-b)}{2}, \qquad \gamma = \frac{d-a}{2}, \qquad \delta = \frac{i(d+a)}{2}.$$

*There is a one-to-one correspondence between the paraboloids*

$$F = ast + bs + ct + d$$

*of the space* $S_3$ *and the points of a four-dimensional space. The addition of*

$$ast + bs + ct + d$$

*to the function* $F$ *in* (6) *has the effect of translating the minimal* $M_3$ *to the point* $(\alpha, \beta, \gamma, \delta)$. *Finally, by virtue of equations* (5) *there is established a one-to-one correspondence between the* $\infty^5$ *minimal lines of* $S_4$ *and the* $\infty^5$ *surface-elements* $F$, $s$, $t$, $F'_s$, $F'_t$ *of* $S_3$. This correspondence will be discussed in Sections 7 and 8 of this paper.

The system (6) defines an isotropic congruence of lines in four-dimensional space. If we interpret the quantities $dX_1/dX_4$, $dX_2/dX_4$, $dX_3/dX_4$ as coördinates in space at infinity, we see at once that any one of these lines meets the imaginary sphere at infinity in a point; hence we may consider the quantities

$$(10) \qquad \frac{-i(s+t)}{1+st}, \quad \frac{t-s}{1+st}, \quad \frac{-i(st-1)}{1+st}$$

as coördinates of points on this sphere. If we have an isotropic congruence, the quantities $F$, $F'_s$ and $F'_t$ are fixed whenever $s$ and $t$ are; hence, *to every point on the sphere at infinity corresponds a minimal line of the congruence.*

## § 2. *The focal surface.*

We may obtain a focal surface of the congruence (6) in the following manner. We choose a generator $D$ of the congruence and pass through it a developable surface of two dimensions, an $M_2$, whose generators belong to the congruence. If $s_0$ and $t_0$ are the parameters belonging to the generator $D$, it is clear that we must find a curve determined by a relation $t = \phi(s)$ such that the generators are tangent to it and moreover such that $t_0 = \phi(s_0)$. The curve will then be the edge of regression of the minimal developable $M_2$. Writing the condition that two consecutive generators intersect, we have in addition to the three equations (5) the following three:

$$
(11) \qquad \begin{aligned} (X_1 + iX_2)dt + dF - s dF'_s - F'_s ds &= 0, \\ (X_3 + iX_4)dt + dF'_s &= 0, \\ (X_3 + iX_4)ds + dF'_t &= 0. \end{aligned}
$$

If from the six equations (5) and (11) we eliminate $X_1$, $X_2$, $X_3$, $X_4$, we obtain the two relations

$$
dF - F'_s ds - F'_t dt = 0,
$$

$$
F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0.
$$

The first relation is obviously satisfied for any isotropic congruence provided $F(s, t)$ is finite and continuous and possesses finite and continuous derivatives. The second condition shows that the edge of regression is determined as either of the two solutions of the differential equation

$$
(12) \qquad F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0
$$

which for $s = s_0$ gives $t_0 = \phi(s_0)$. But such a solution always exists, hence *there exist two minimal developables passing through an arbitrary generator $D$ of the congruence.* In order to obtain the focal surface of the $\infty^1$ minimal developables we substitute the value of $dt$ from (12) in the last of equations (11), so that we have

$$
(13) \qquad X_3 + iX_4 + F''_{st} + \sqrt{F''_{s^2} F''_{t^2}} = 0,
$$

or, using the second value of $ds$,

$$
(13') \qquad X_3 + iX_4 + F''_{st} - \sqrt{F''_{s^2} F''_{t^2}} = 0.
$$

The equations of the focal surface may now be obtained by solving (5) and (13), or (5) and (13′), for $X_1$, $X_2$, $X_3$, and $X_4$. We find

$$
(14) \qquad \begin{aligned} X_1 &= \tfrac{1}{2}[(s+t)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - F'_t - F'_s], \\ X_2 &= -\frac{i}{2}[(s-t)(F''_{st} \mp \sqrt{F''_{s^2}F''_{t^2}}) - F'_t + F'_s], \\ X_3 &= \tfrac{1}{2}[(st-1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F], \\ X_4 &= \frac{i}{2}[(st+1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F]. \end{aligned}
$$

We have thus proved the

THEOREM. *An isotropic congruence of four-dimensional space has a focal surface consisting of two sheets. The edges of regression on the focal surface are minimal curves satisfying the differential equation*

$$
F''_{s^2} ds^2 - F''_{t^2} dt^2 = 0.
$$

*Through each generator of the congruence pass two minimal developables $M_2$ which are tangent to the surface.*

The linear element $d\sigma$ of the surface (14) may be obtained without difficulty. We find

$$d\sigma^2 = -\sqrt{F''_{s^2}F''_{t^2}}(\sqrt{F''_{s^2}}ds - \sqrt{F''_{t^2}}dt)^2 \tag{15}$$

for the focal sheet $S_1$ and

$$d\sigma^2 = \sqrt{F''_{s^2}F''_{t^2}}(\sqrt{F''_{s^2}}ds + \sqrt{F''_{t^2}}dt)^2 \tag{15'}$$

for the second sheet $S_2$. Introducing the minimal curves on $S_1$ as one set of coördinate curves, we may reduce the element (15) or (15′) to the form

$$d\sigma^2 = \lambda^2 d\beta^2,$$

where $\lambda$ is a function of $\beta$ and a second coördinate $\alpha$ which for (15) is such that the expression

$$\frac{i\sqrt[4]{F''_{s^2}F''_{t^2}}(\sqrt{F''_{s^2}}ds - \sqrt{F''_{t^2}}dt)}{\lambda} \tag{16}$$

is an exact differential. Similarly for (15′). The element is thus brought into the same form as that of a minimal developable in ordinary space referred to its minimal curves as one set of coördinate lines. *The focal surface* (14) *is, however, not developable, for the curves $\beta = const.$ are not minimal straight lines, but curves.* To show this it is only necessary to take a single example. We put $F = s^2t^2$ and form the corresponding equations of the focal sheets:

$$(17)\qquad S_1:\ \begin{aligned} X_1 &= 2st(s+t), \\ X_2 &= -2ist(s-t), \\ X_3 &= \tfrac{3}{2}st(st-2), \\ X_4 &= \tfrac{3}{2}ist(st+2). \end{aligned} \qquad S_2:\ \begin{aligned} X_1 &= 0, \\ X_2 &= 0, \\ X_3 &= -\tfrac{1}{2}st(st+2), \\ X_4 &= -\frac{i}{2}st(st-2). \end{aligned}$$

The linear elements of these surfaces are respectively

$$d\sigma_1^2 = -[2\sqrt{st}\,t\,ds - 2\sqrt{st}\,s\,dt]^2,$$

$$d\sigma_2^2 = [2\sqrt{st}\,t\,ds + 2\sqrt{st}\,s\,dt]^2.$$

If we consider the focal sheet $S_1$, we have

$$\lambda = 2i\sqrt{st}\,s\,t,$$

so that we get

$$\frac{ds}{s} - \frac{dt}{t} = d\beta.$$

Integrating and putting $e^\beta$ equal to a new $\beta$, we have

$$\frac{s}{t} = \beta.$$

Choosing the lines $t$ for the second set of coördinates, the equations for $S_1$ become

$$X_1 = 2t^3\beta(1+\beta), \qquad X_2 = -2it^3\beta(\beta-1),$$
$$X_3 = \tfrac{3}{2}t^2\beta(t^2\beta-2), \qquad X_4 = \tfrac{3}{2}it^2\beta(t^2\beta+2).$$

The curves $\beta = $ const. are minimal curves, since

$$\Sigma\left(\frac{\partial X_i}{\partial t}\right)^2 = 0;$$

it is also readily verified that

$$\Sigma\frac{\partial X_i}{\partial t}\cdot\frac{\partial X_i}{\partial \beta} = 0.$$

The second sheet $S_2$ reduces to a curve in the plane $X_1=0$, $X_2=0$. We have

$$\frac{ds}{s} + \frac{dt}{t} = d\beta,$$

hence, putting $st = \beta$ the equations become

$$X_1 = 0, \qquad X_2 = 0, \qquad X_3 = -\tfrac{1}{2}\beta(\beta+2), \qquad X_4 = \frac{-i}{2}\beta(\beta-2).$$

§ 3. *The linear element.*

In ordinary space a ruled surface whose generators are minimal lines is necessarily a minimal developable and its linear element is a perfect square. It is different in four-dimensional space. A ruled surface with minimal lines as generators is not in general a minimal developable. Let there be given a ruled surface

$$X_1 = f_1\alpha + \phi_1, \qquad X_2 = f_2\alpha + \phi_2,$$
$$X_3 = f_3\alpha + \phi_3, \qquad X_4 = f_4\alpha + \phi_4,$$

and let us suppose that $\beta = $ const. are minimal lines, in which case we must have

(18) $$f_1^2 + f_2^2 + f_3^2 + f_4^2 = 0.$$

If this surface is a minimal developable, two consecutive lines on the surface must intersect, hence we must have

$$X_1 = (f_1 + df_1)\alpha + \phi_1 + d\phi_1 = f_1\alpha + \phi_1,$$
$$X_2 = (f_2 + df_2)\alpha + \phi_2 + d\phi_2 = f_2\alpha + \phi_2,$$
$$X_3 = (f_3 + df_3)\alpha + \phi_3 + d\phi_3 = f_3\alpha + \phi_3,$$
$$X_4 = (f_4 + df_4)\alpha + \phi_4 + d\phi_4 = f_4\alpha + \phi_4,$$

from which we get

$$\alpha f'_1 + \phi'_1 = 0, \qquad \alpha f'_2 + \phi'_2 = 0, \qquad \alpha f'_3 + \phi'_3 = 0, \qquad \alpha f'_4 + \phi'_4 = 0,$$

or,

$$\frac{\phi'_1}{f'_1} = \frac{\phi'_2}{f'_2} = \frac{\phi'_3}{f'_3} = \frac{\phi'_4}{f'_4}, \tag{19}$$

which is the condition that the ruled surface shall be developable.

*The linear element is now a perfect square.* In fact, taking account of the relation (19) and the identities

$$\sum_1^4 f_i^2 = 0, \qquad \sum_1^4 f_i f''_i = 0, \qquad \sum_1^4 f_i \phi'_i = 0,$$

we have

$$ds^2 = \sum_1^4 (\alpha f'_i + \phi'_i)^2 d\beta^2 = \left(\alpha + \frac{\phi'_4}{f'_4}\right)^2 \Sigma f'^2_i \cdot d\beta^2.$$

The edge of regression is

$$X_1 = -f_1 \frac{\phi'_1}{f'_1} + \phi_1, \qquad X_2 = -f_2 \frac{\phi'_2}{f'_2} + \phi_2,$$

$$X_3 = -f_3 \frac{\phi'_3}{f'_3} + \phi_3, \qquad X_4 = -f_4 \frac{\phi'_4}{f'_4} + \phi_4,$$

which is seen to be a minimal curve, if account is taken of the conditions (18) and (19). We have thus proved the

THEOREM. *If a ruled surface in four-dimensional space whose generators are minimal lines is a developable surface, its linear element is a perfect square.*

We have already seen that the converse of this theorem is not true, since the focal surface of an isotropic congruence is not developable, although its element is a perfect square.

We shall now prove the

THEOREM. *If a surface in four-dimensional space has its linear element a perfect square, it is either a minimal developable, or the focal surface of* $\infty^1$ *such developables.*

Let the linear element be written

$$ds^2 = (\sqrt{E}\,du + \sqrt{G}\,dv)^2.$$

If we put

$$\frac{\sqrt{E}\,du + \sqrt{G}\,dv}{\lambda} = d\beta,$$

where $\lambda$ is a function of $u$ and $v$ which will make the left side of the equation an exact differential, the linear element takes the form

$$ds^2 = \lambda^2 d\beta^2. \tag{20}$$

There exist therefore on the surface $\infty^1$ minimal curves $\beta = \text{const.}$ (a single family it should be noticed) which may serve as one set of lines of reference, an arbitrary system $\alpha = \text{const.}$ being chosen as the second set. A minimal curve $\beta = \text{const.}$ determines a minimal developable whose edge of regression is the minimal curve; *this minimal developable will touch the surface along a curve whose direction is conjugate* to the curve $\beta = const.$* In fact, we have from (20),

$$\Sigma\left(\frac{\partial X_i}{\partial \alpha}\right)^2 = 0, \qquad \Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial X_i}{\partial \beta} = 0. \tag{21}$$

Differentiating the first with respect to $\alpha$ and $\beta$, we have

$$\Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial^2 X_i}{\partial \alpha^2} = 0, \qquad \Sigma \frac{\partial X_i}{\partial \alpha}\frac{\partial^2 X_i}{\partial \alpha \partial \beta} = 0. \tag{22}$$

From (21) and (22) it follows that the determinant

$$e = \left| \frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha^2}, \frac{\partial^2 X_4}{\partial \alpha \partial \beta} \right|$$

vanishes. But this is precisely the condition that two consecutive tangent planes at $(\alpha, \beta)$ and $(\alpha + d\alpha, \beta)$ shall intersect in a line. The lines $\beta$ and the lines whose direction is the direction of the line of intersection of the two tangent planes are therefore conjugate lines. Since the system $\alpha = \text{const.}$ was arbitrary, we may introduce this conjugate system as our lines of reference. The new lines $\alpha'$ will then be the solution of the differential equation

$$2f d\alpha + g d\beta = 0,$$

where $f$ and $g$ are the determinants

$$\left| \frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha^2}, \frac{\partial^2 X_4}{\partial \beta^2} \right|, \qquad \left| \frac{\partial X_1}{\partial \alpha}, \frac{\partial X_2}{\partial \beta}, \frac{\partial^2 X_3}{\partial \alpha \partial \beta}, \frac{\partial^2 X_4}{\partial \beta^2} \right|.$$

* There exists on a surface $M_2$ in $S_4$ a double family of curves which have been called asymptotic lines by K. KOMMERELL. They form a unique system of conjugate lines, that is to say, if we pass along an asymptotic line of the first set with a tangent plane to the surface, any two consecutive planes will intersect in a line whose direction is the direction of an asymptotic line of the second set. (It should be remembered that in $S_4$ planes intersect in general in a point.) In ordinary space there exists at any point on a surface an involution of such lines passing through the point; in $S_4$ there are only two conjugate directions at any point on an $M_2$. See KOMMERELL, Thesis, pp. 30 and 31. Also a paper by the same author, *Riemannsche Flächen in ebenem Raum von vier Dimensionen*, Mathematische Annalen, vol. 60 (1905), pp. 548–596. KOMMERELL's thesis is entitled: *Die Krümmung der zweidimensionalen Gebilde in ebenem Raum von vier Dimensionen*, Thesis, Tübingen, 1897.

The differential equation of these asymptotic-conjugate lines is:

$$e d\alpha^2 + 2f d\alpha\, d\beta + g d\beta^2 = 0,$$

where $e$, $f$ and $g$ are the determinants mentioned above.

The surface being thus referred to its unique system of conjugate-asymptotic lines, it follows from the property of such a system that if we pass along a curve $\alpha = \text{const.}$, the consecutive tangent planes will intersect in minimal tangent lines whose directions are those of the minimal curves $\beta = \text{const.}$ The surface is therefore the focal surface of the $\infty^2$ minimal lines determined by the curves $\beta = \text{const.}$ If the curves $\beta = \text{const.}$ are straight lines, the surface is a minimal developable.

Consider now the focal surface (14) and a generator $D_1$ tangent to the minimal curves $C_1$ and $C_2$ on the two sheets $S_1$ and $S_2$ respectively. Let $T_1$ and $T_2$ be the curves conjugate to $C_1$ and $C_2$ respectively and passing through the focal points $F_1$ and $F_2$. The tangent planes at consecutive points on an edge of regression intersect in a line which must be tangent to the conjugate curve $T_2$ at $F_2$. Likewise, tangent planes at $F_2$ and at a consecutive point on $T_2$ intersect in a line which is $D_1$. Hence $D_1$ and the tangent to $T_2$ are conjugate and asymptotic directions on $S_2$ at $F_2$. The same will be true on $S_1$ for the curves $C_1$ and $T_1$.

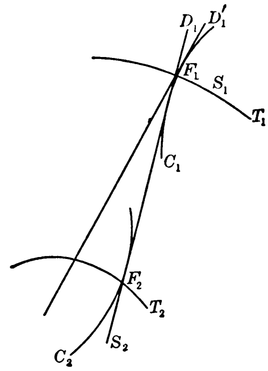

FIG. 1.

If the curves $C_2$ satisfy the differential equation

$$\sqrt{F''_{s^2}}\,ds - \sqrt{F''_{t^2}}\,dt = 0, \tag{23}$$

the curves $T_2$ will satisfy the equation

$$\sqrt{F''_{s^2}}\,ds + \sqrt{F''_{t^2}}\,dt = 0. \tag{24}$$

Again, the equation (24) will be satisfied by the curves $C_1$ on $S_1$, that is, by the edges of regression, while the curves $T_1$ will satisfy the equation (23). We have thus proved the

THEOREM. *On each sheet of the focal surface the family of lines whose differential equation is*

$$F''_{s^2}ds^2 - F''_{t^2}dt^2 = 0$$

*constitutes a double set of conjugate and asymptotic lines.*

## § 4. *The middle surface.*

The generator $D$ which determines a pair of minimal developables is a common tangent to the two edges of regression on $S_1$ and $S_2$ respectively. Suppose now we take the mid-points of all the segments $\overline{F_1F_2}$, $F_1$ and $F_2$ being the focal points. We obtain a new locus which we shall call the *middle surface* of the congruence. From equations (14) we find that the equations of this surface are

$$X_1 = \tfrac{1}{2}[(s+t)F''_{st} - F'_t - F'_s],$$

$$
(25) \qquad
\begin{aligned}
X_2 &= -\frac{i}{2}\left[(s-t)F''_{st} - F'_t + F'_s\right],\\
X_3 &= \tfrac{1}{2}\left[(st-1)F''_{st} - tF'_t - sF'_s + F\right],\\
X_4 &= \frac{i}{2}\left[(st+1)F''_{st} - tF'_t - sF'_s + F\right].
\end{aligned}
$$

*The lines $s$ and $t$ on the surface are minimal lines.* In fact, if we calculate the linear element $dS$ we find that

$$\Sigma\left(\frac{\partial X_i}{\partial s}\right)^2 \quad \text{and} \quad \Sigma\left(\frac{\partial X_i}{\partial t}\right)^2$$

vanish and the following simple form results

$$(26) \qquad dS^2 = F''_{t^2} F''_{s^2}\, ds\, dt.$$

We shall next prove the following

THEOREM. *The surface* (25) *is the envelope of the* $\infty^2$ *minimal spaces,*

$$(27) \quad (s+t)X_1 + i(t-s)X_2 + (st-1)X_3 + i(st+1)X_4 + F = 0.$$

In fact, this space passes through the four vertices $(s, t)$; $(s+ds, t)$; $(s, t+dt)$; $(s+ds, t+dt)$ of the elementary quadrilateral formed by the coördinate lines $s$ and $t$. From (25) we have

$$
(28) \qquad
\begin{aligned}
&(s+t)\frac{\partial X_1}{\partial s} + i(t-s)\frac{\partial X_2}{\partial s} + (st-1)\frac{\partial X_3}{\partial s} + i(st+1)\frac{\partial X_4}{\partial s} = 0,\\
&(s+t)\frac{\partial X_1}{\partial t} + i(t-s)\frac{\partial X_2}{\partial t} + (st-1)\frac{\partial X_3}{\partial t} + i(st+1)\frac{\partial X_4}{\partial t} = 0,\\
&(s+t)\frac{\partial^2 X_1}{\partial s\,\partial t} + i(t-s)\frac{\partial^2 X_2}{\partial s\,\partial t} + (st-1)\frac{\partial^2 X_3}{\partial s\,\partial t} + i(st+1)\frac{\partial^2 X_4}{\partial s\,\partial t} = 0,
\end{aligned}
$$

which proves the statement. The minimal lines of an $M_3$ being given by the equations

$$
(29) \qquad
\begin{aligned}
(s+t)X_1 + i(t-s)X_2 + (st-1)X_3 + i(st+1)X_4 + F &= 0,\\
X_1 - iX_2 + t(X_3 + iX_4) + F'_s &= 0,\\
X_1 + iX_2 + s(X_3 + iX_4) + F'_t &= 0,
\end{aligned}
$$

the middle surface may be obtained by adding to this set the minimal space

$$(30) \qquad X_3 + iX_4 + F''_{st} = 0$$

and solving for $X_1$, $X_2$, $X_3$ and $X_4$. The minimal space (30) is obtained by differentiating either the second of (29) with respect to $s$, or the third with respect to $t$.

If we choose $F$ as a solution of the differential equation

$$(31) \qquad X_4 = \frac{i}{2}\left[(st+1)F''_{st} - tF'_t - sF'_s + F\right] = 0,$$

we shall obtain a surface situated in ordinary space ($X_1$, $X_2$, $X_3$). The general solution of (31) is

$$(32) \qquad F = 2tS + 2sT - (1+st)(S' + T'),$$

where $S$ and $T$ are functions of $s$ and $t$ respectively. Introducing this value of $F$ in equations (25) we find the following equations

$$(33) \qquad \begin{aligned} X_1 &= \tfrac{1}{2}(1-s^2)S'' + sS' - S + \tfrac{1}{2}(1-t^2)T'' + tT' - T, \\ X_2 &= \frac{i}{2}(1+s^2)S'' - isS' + iS - \frac{i}{2}(1+t^2)T'' + itT' - iT, \\ X_3 &= sS - S' + tT'' - T', \\ X_4 &= 0, \end{aligned}$$

which we recognize at once as the most general equations of a minimal surface in the so-called Weierstrassian form. The corresponding focal surface is obtained from (14) by substituting for $F$ the value obtained above. We have then, calling the coördinates of the focal surface $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$, $\bar{X}_4$ and those of the middle surface $X_1$, $X_2$, $X_3$, $X_4$, ($X_4 = 0$),

$$(34) \qquad \begin{aligned} \bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)(1+st)\sqrt{S'''T'''}, \\ \bar{X}_4 &= \pm \tfrac{1}{2}(st+1)^2\sqrt{S'''T'''}, \end{aligned}$$

where the upper sign goes with the focal sheet $S_1$ and the lower with $S_2$. From these equations it appears that the focal surface may be obtained in the following manner when the minimal surface is given: We construct in ordinary space, $X_4 = 0$, the point ($X_1$, $X_2$, $X_3$) on the minimal surface; from this point we lay off a distance $iR$, where $R$ is one of the principal radii of curvature of the minimal surface at the point ($X_1$, $X_2$, $X_3$), parallel to the $X_4$-axis; the point thus found is a point on the focal sheet $S_1$, if the upper sign is used. If the lower sign is used, we obtain a point on $S_2$ which is symmetrical to the first, the center of symmetry being the point $s$, $t$ on the minimal surface. That $\bar{X}_4 = \pm iR$ may be verified in the usual manner. (See Darboux, *Théorie générale des surfaces*, vol. 1, p. 302, equations (15).) We may therefore say:

*A minimal surface in ordinary space is the middle surface of a certain minimal congruence of four-dimensional space*, viz., the congruence for which $F$ is defined by equations (32).

This new definition of an ordinary minimal surface suggests a definition for a minimal surface in four-dimensional space. The usual definition fails when extended to higher space, since two surfaces meet usually in discrete points.*

*Definition. A minimal surface in four-dimensional space is the middle surface of an isotropic congruence.*

The most general equations of such a surface referred to its minimal curves are given by (25). They include ordinary minimal surfaces in three-dimensional space, namely when $F$ is a solution of any one of the four differential equations

$$(35)\qquad \begin{aligned} (s+t)F''_{st} - F'_t - F'_s &= 0, \\ (s-t)F''_{st} - F'_t + F'_s &= 0, \\ (st-1)F''_{st} - tF'_t - sF'_s + F &= 0, \\ (st+1)F''_{st} - tF'_t - sF'_s + F &= 0. \end{aligned}$$

Either one of the four values of $F$ introduced in (25) will give the equations of an ordinary minimal surface in the most general form. The solution of the last equation will furnish, as we have seen, the Weierstrassian form; the other forms may be deduced from this by the usual substitutions employed in the theory of minimal surfaces.

## § 5. *Orthogonal projections.*

We shall put the equations of the focal surface (14) in the following form:

$$(36)\qquad \begin{aligned} \bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)\sqrt{F''_{s^2}F''_{t^2}}, \\ \bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)\sqrt{F''_{s^2}F''_{t^2}}, \\ \bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)\sqrt{F''_{s^2}F''_{t^2}}, \\ \bar{X}_4 &= X_4 \pm \frac{i}{2}(st+1)\sqrt{F''_{s^2}F''_{t^2}}, \end{aligned}$$

where $X_i$ are the coördinates of the corresponding middle or minimal surface. The following relation between the coördinates $\bar{X}_i$ and $X_i$ is seen to hold:

$$(37)\qquad (\bar{X}_1 - X_1)^2 + (\bar{X}_2 - X_2)^2 + (\bar{X}_3 - X_3)^2 + (\bar{X}_4 - X_4)^2 = 0.$$

We shall now make use of a transformation which has been employed by Lie in the case of $n = 3$.† Let $(X_1, X_2, X_3, X_4)$ be the vertex of a minimal

* See KOMMERELL, loc. cit., p. 588.

† See LIE-SCHEFFERS, loc. cit., pp. 428–429.

cone in $S_4$ and $(x_1, x_2, x_3, x_4)$ the running coördinates. The equation of the cone is then

$$(38) \qquad (x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2 + (x_4 - X_4)^2 = 0.$$

If we cut this cone by the space $x_4 = 0$, we obtain the sphere in three-dimensional space $S_3$

$$(39) \qquad (x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_4)^2 + X_4^2 = 0.$$

We now establish a correspondence between the points of the space $(X_1, X_2, X_3, X_4)$ and the spheres in $S_3$ by means of the transformation

$$(40) \qquad x_1 = X_1, \qquad x_2 = X_2, \qquad x_3 = X_3, \qquad R = iX_4,$$

such that the vertex of the minimal cone is projected on $X_4 = 0$ into the center of the sphere, the radius being equal to $R = iX_4$. To every point $(X_1, X_2, X_3, X_4)$ corresponds a sphere of radius $R$ and center $(x_1, x_2, x_3, 0)$, and conversely, to every sphere in the space $x_4 = 0$ with center $(x_1, x_2, x_3)$ and radius $R$ there correspond the two points

$$(41) \qquad X_1 = x_1, \qquad X_2 = x_2, \qquad X_3 = x_3, \qquad X_4 = \pm iR$$

in $S_4$. If two points $(X_1, X_2, X_3, X_4)$ and $(X'_1, X'_2, X'_3, X'_4)$ lie in a minimal line, the corresponding minimal cones are tangent to each other, and the corresponding spheres in $S_3$ will also be tangent to each other, since

$$(X_1 - X'_1)^2 + (X_2 - X'_2)^2 + (X_3 - X'_3)^2 = (R - R')^2.$$

To all the points of a minimal line correspond $\infty^1$ spheres which touch at the point of intersection of the minimal line with the space $X_4 = 0$. This means that the spheres have a surface-element in common at that point.

Consider now a flat space $E$; it intersects the space at infinity in a plane. The pole of this plane with respect to the (imaginary) sphere at infinity is the point at infinity which determines the direction of the $\infty^3$ normals to $E$. If this pole lies on the sphere, $\infty^2$ of these normals will lie in $E$ and $E$ is a minimal space.

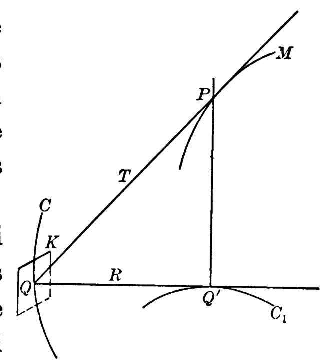


FIG. 2.

The tangents to a minimal curve form a minimal developable. Let $T$ be any one of these tangents and $P$ the point of contact; the intersection of the space $X_4 = 0$ with this developable will be a twisted curve $C$ of that space. We shall now project the space $S_4$ orthogonally on the space $X_4 = 0$. Let $Q$ be the point of intersection of the minimal line $T$ with $X_4 = 0$, and let $K$, the intersection of $X_4 = 0$ with the minimal space $E$ determined by $T$, be the plane of the surface element at $Q$. Since this plane is contained in $E$, the line $T$, being normal to $E$, must be normal to every line and plane in $E$; hence $T$ is normal to $K$. The projection of $T$ on $X_4$, or $\overline{QQ'}$, must also be perpendicular to $K$. Hence, when the point $P$ moves along the minimal curve, the projected point $Q'$ moves along the curve $C'$ which is an evolute of $C$. The line $C$ is therefore a line of curvature on the surface obtained by intersecting the isotropic congruence with the space $X_4 = 0$. Hence the

THEOREM. *The orthogonal projection of a minimal curve $M$ on the space $X_4 = 0$ is an evolute of the curve of intersection of the developable of $M$ with that space.*

We have seen that a congruence of minimal lines determines a double family of minimal developables and that it has a focal surface which in general consists of two sheets on which are situated the $\infty^1$ edges of regression of the developables. If now this entire congruence be projected orthogonally, we obtain curves and surfaces which we shall study in detail. It will be convenient to use the analytical method, although a purely geometric method might also be applied.

If we cut the minimal congruence (6), § 1, by the space $X_4 = 0$, we obtain a surface in three-dimensional space whose coördinates we shall denote by $x_1$, $x_2$, and $x_3$. We have then

$$(42)\qquad \begin{aligned} x_1 &= -\frac{(s+t)F + (1-s^2)F'_s + (1-t^2)F'_t}{2(1+st)}, \\ x_2 &= i\frac{(s-t)F - (1+s^2)F'_s + (1+t^2)F'_t}{2(1+st)}, \\ x_3 &= \frac{F - sF'_s - tF'_t}{1+st}. \end{aligned}$$

We shall call this surface the *surface of reference* of the congruence. We see at once that the equations (42) are those of any surface in $S_3$ referred to the curves of contact of tangent cones which have their vertices on the imaginary circle at infinity; these curves ($s$ and $t$) have for spherical images the rectilinear generators of a sphere. (Compare DARBOUX, l. c., vol. 1, p. 243, and equations (33), p. 246.)

To the minimal curves on the focal surface of the congruence correspond lines of curvature on the surface of reference. In fact, the differential equation of the minimal curves is

$$(12)\qquad F''_{s^2}ds^2 - F''_{t^2}dt^2 = 0,$$

which is also the differential equation of the lines of curvature on the surface (42). The directions of these lines at any point $s$, $t$ are the bisectors of the angles between the coördinate curves.*

*See DARBOUX, l. c., vol. 1, p. 243.

If now we project the focal sheets orthogonally on the space $X_4 = 0$ by means of the transformation (40), we obtain the equations

$$
(43)\qquad
\begin{aligned}
\bar{x}_1 &= \tfrac{1}{2}[(s+t)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - F'_t - F'_s],\\
\bar{x}_2 &= -\frac{i}{2}[(s-t)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - F'_t + F'_s],\\
\bar{x}_3 &= \tfrac{1}{2}[(st-1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F],\\
R &= -\tfrac{1}{2}[(st+1)(F''_{st} \pm \sqrt{F''_{s^2}F''_{t^2}}) - tF'_t - sF'_s + F].
\end{aligned}
$$

The first three equations are seen to be the equations of the surface of centers of the surface of reference, while the last equation gives the values of the principal radii of curvature of the surface. *The minimal curves on the focal sheets in $S_4$ become geodesics on the surface of centers* (by the theorem on p. 416). The isotropic congruence becomes the normal congruence with respect to the lines of curvature of the surface of reference. The middle surface of the minimal congruence becomes the middle surface of the normal congruence.

If, in particular, $F$ is a solution of the equation

$$(st+1)F''_{st} - tF'_t - sF'_s + F = 0,$$

the system (43) reduces to (34), while the value of $R$ is

$$R = \pm \tfrac{1}{2}(1+st)^2\sqrt{S'''T'''},$$

which is the expression for the principal radii of curvature. We notice that they are numerically equal with opposite signs; hence, the middle surface is an ordinary minimal surface, as was proved before on p. 413, and it is identical with the surface of reference of the minimal congruence. Hence the

THEOREM. *If the middle surface of a minimal congruence in four-dimensional space is a minimal surface in ordinary space, it is identical with the surface of reference.*

The two conjugate-asymptotic directions on the focal surface in $S_4$ become conjugate directions on the surface of centers, the geodesics being one family (corresponding to the minimal curves) and the second, the curves of contact of the developable surfaces determined by the second family of lines of curvature on the surface of reference. In this particular case, then, conjugate directions are preserved when the surface is projected orthogonally on $X_4 = 0$. As an example we shall take the case $F = s^2t^2$ used before. The focal surface (14) becomes, when projected orthogonally on to the space $(x_1, x_2, x_3)$,

$$
\begin{aligned}
\bar{x}_1 &= 2\alpha\sqrt{\frac{\alpha}{\beta}}(1+\beta), &\qquad \bar{x}_3 &= -2i\alpha\sqrt{\frac{\alpha}{\beta}}(\beta-1),\\
\bar{x}_3 &= \tfrac{3}{2}\alpha(\alpha-2), & iR &= \tfrac{3}{2}i\alpha(\alpha+2).
\end{aligned}
$$

Since the lines $\beta = \text{const.}$ are minimal curves on the focal surface (14), they are geodesics on the surface of centers. The lines $\alpha = \text{const.}$ are conjugate to these, since the expression

$$HD' = \begin{vmatrix} \frac{\partial \bar{x}_2}{\partial \alpha} & \frac{\partial \bar{x}_3}{\partial \alpha} \\ \frac{\partial \bar{x}_2}{\partial \beta} & \frac{\partial \bar{x}_3}{\partial \beta} \end{vmatrix} \frac{\partial^2 \bar{x}_1}{\partial \alpha \partial \beta} + \begin{vmatrix} \frac{\partial \bar{x}_3}{\partial \alpha} & \frac{\partial \bar{x}_1}{\partial \alpha} \\ \frac{\partial \bar{x}_3}{\partial \beta} & \frac{\partial \bar{x}_1}{\partial \beta} \end{vmatrix} \frac{\partial^2 \bar{x}_2}{\partial \alpha \partial \beta} + \begin{vmatrix} \frac{\partial \bar{x}_1}{\partial \alpha} & \frac{\partial \bar{x}_2}{\partial \alpha} \\ \frac{\partial \bar{x}_1}{\partial \beta} & \frac{\partial \bar{x}_2}{\partial \beta} \end{vmatrix} \frac{\partial^2 \bar{x}_3}{\partial \alpha \partial \beta}$$

vanishes identically, as a simple calculation will show.* The surface of reference is

$$x_1 = \frac{(\beta + 1)(\alpha - 2)\sqrt{\alpha}\,\alpha}{2(1 + \alpha)\sqrt{\beta}}, \qquad x_2 = \frac{i(1 - \beta)(\alpha - 2)\sqrt{\alpha}\,\alpha}{2(1 + \alpha)\sqrt{\alpha}}, \qquad x_3 = \frac{-3\alpha^2}{1 + \alpha},$$

where $\alpha$ and $\beta$ are lines of curvature. This surface, though written in imaginary form, is real; in fact, writing the surface in the form

$$x_1 = -\frac{st(s + t)(2 - st)}{2(1 + st)}, \qquad x_2 = \frac{ist(t - s)(st - 2)}{2(1 + st)}, \qquad x_3 = \frac{3s^2 t^2}{1 + st},$$

we may obtain a real form by making $s$ and $t$ conjugate imaginaries; thus, suppose we put

$$s = p + iq, \qquad t = p - iq,$$

then

$$x_1 = -\frac{2(p^2 + q^2)(2 - p^2 - q^2)p}{2(1 + p^2 + q^2)}, \qquad x_2 = \frac{2(p^2 + q^2)(p^2 + q^2 - 2)q}{2(1 + p^2 + q^2)},$$

$$x_3 = \frac{-3(p^2 + q^2)^2}{1 + p^2 + q^2}.$$

All the focal surfaces (14) in $S_4$ are imaginary, as are also the minimal developables belonging to a focal surface. The surface of reference, as well as the surface of centers and middle surface, will in general have real sheets which are obtained by making $s$ and $t$ conjugate variables. The middle surface of a minimal congruence will have real sheets only if it lies in a three-dimensional space, in which case it is an ordinary minimal surface.

## § 6. *Translation surfaces in $S_4$.*

We shall now consider the case where the middle surface, or "minimal" surface of the congruence is a translation-surface. The most general solution of the differential equation

---

* The second focal sheet being a focal line, the second sheet of the surface of centers is also a line. DARBOUX has discussed such surfaces, l. c., vol. 1, p. 226. The surface of reference in this case is the envelope of a sphere of variable radius whose center describes the curve $\bar{x}_1 = 0$, $\bar{x}_2 = 0$, $\bar{x}_3 = -\alpha/2(\alpha + 2)$, the radius being $R = -\alpha/2(\alpha - 2)$; the surface is a canal surface.

$$(44)\qquad X_4 = \frac{i}{2}\,[(st+1)F''_{st} - tF'_t - sF'_t + F] = \frac{i}{2}(S+T),$$

where $S$ and $T$ are two arbitrary functions of $s$ and $t$ respectively, will give a value of $F$ such that the corresponding surface is one of translation. If we put $F''_{st} = S' + T'$, the left side of (44) becomes

$$(45)\qquad X_4 = \frac{i}{2}\,[S_1 + T_1 - tT'_1 - sS'_1 - S' - T'],$$

which is of the form $S + T$. Introducing the function $F$ obtained by integrating the equation

$$F''_{st} = S' + T',$$

that is to say,

$$(46)\qquad F = tS + sT + S_1 + T_1$$

in the system (25), we obtain expressions similar to (45) for the remaining coördinates, viz:

$$(47)\qquad \begin{aligned} X_1 &= \tfrac{1}{2}[sS' - S - S'_1] + \tfrac{1}{2}[tT' - T - T'_1],\\ X_2 &= -\frac{i}{2}[sS' - S + S'_1] + \frac{i}{2}[tT' - T + T'_1],\\ X_3 &= -\tfrac{1}{2}[sS'_1 + S' - S_1] - \tfrac{1}{2}[tT'_1 + T' - T_1],\\ X_4 &= -\frac{i}{2}[sS'_1 - S' - S_1] - \frac{i}{2}[tT'_1 - T' - T_1], \end{aligned}$$

which may be written in the following form

$$(47')\qquad \begin{aligned} X_1 &= \tfrac{1}{2}\int(sS'' - S''_1)\,ds + \tfrac{1}{2}\int(tT'' - T''_1)\,dt,\\ X_2 &= -\frac{i}{2}\int(sS'' + S''_1)\,ds + \frac{i}{2}\int(tT'' + T''_1)\,dt,\\ X_3 &= -\tfrac{1}{2}\int(sS''_1 + S'')\,ds - \tfrac{1}{2}\int(tT''_1 + T'')\,dt,\\ X_4 &= -\frac{i}{2}\int(sS''_1 - S'')\,ds - \frac{i}{2}\int(tT''_1 - T'')\,dt. \end{aligned}$$

The focal surface may now be obtained by substituting the value of $F$ in equations (36). We find, denoting the coördinates by $\bar{X}_i$ and those of the middle surface by $X_i$,

$$(48)\qquad \begin{aligned} \bar{X}_1 &= X_1 \pm \tfrac{1}{2}(s+t)\sqrt{(tS'' + S''_1)(sT'' + T''_1)},\\ \bar{X}_2 &= X_2 \mp \frac{i}{2}(s-t)\sqrt{(tS'' + S''_1)(sT'' + T''_1)},\\ \bar{X}_3 &= X_3 \pm \tfrac{1}{2}(st-1)\sqrt{(tS'' + S''_1)(sT'' + T''_1)}, \end{aligned}$$

$$\bar{X}_4 = X_4 \pm \frac{i}{2}(st+1)\sqrt{(tS''+S_1'')(sT''+T_1'')},$$

which is not a translation-surface.

If, in particular, we put $S = 2\bar{S} - s\bar{S}'$, $T = 2\bar{T} - t\bar{T}'$, $S_1 = -\bar{S}'$, $T_1 = -\bar{T}'$, we obtain the ordinary minimal surface (33). From the form of equations (47) we see that these translation-surfaces are imaginary. Since the equations

$$(49)\qquad \begin{aligned} \bar{X}_1 &= sS' - S - S_1', & \bar{X}_2 &= -i(sS' - S + S_1'),\\ \bar{X}_3 &= -(sS_1' + S' - S_1), & \bar{X}_4 &= -i(sS_1' - S' - S_1) \end{aligned}$$

give the most general parametric representation of a minimal curve in $S_4$, it would seem that (47) represents the most general translation-surface whose translation curves are minimal, and that they are therefore all imaginary. This is, however, not so. In fact, a real translation-surface may be obtained in the following manner: Consider two curves like (49); the locus of the middle points of all the chords joining the two curves will be a translation surface (47), that is, we have

$$(50)\qquad \begin{aligned} X_1 &= \tfrac{1}{2}(\bar{X}_1 + \bar{\bar{X}}_1), & X_2 &= \tfrac{1}{2}(\bar{X}_2 + \bar{\bar{X}}_2),\\ X_3 &= \tfrac{1}{2}(\bar{X}_3 + \bar{\bar{X}}_4), & X_4 &= \tfrac{1}{2}(\bar{X}_4 + \bar{\bar{X}}_4), \end{aligned}$$

$\bar{X}_i$ being the coördinates in (49) and $\bar{\bar{X}}_i$ the coördinates of the curve

$$(51)\qquad \begin{aligned} \bar{\bar{X}}_1 &= tT' - T - T_1', & \bar{\bar{X}}_2 &= i(tT' - T + T_1'),\\ \bar{\bar{X}}_3 &= -(tT_1' + T' - T_1), & \bar{\bar{X}}_4 &= -i(tT_1' - T' - T_1). \end{aligned}$$

The surface (47) has no real sheet; if, however, we project the curve (49) by means of the transformation $\bar{X}_1 = \bar{X}_1$, $\bar{X}_2 = \bar{X}_2$, $\bar{X}_3 = \bar{X}_3$, $\bar{X}_4 = -\bar{X}_4$, i. e., if we reflect the curve with respect to the space $\bar{X}_4 = 0$, we get the curve

$$(50')\qquad \begin{aligned} \bar{X}_1 &= sS' - S - S_1',\\ \bar{X}_2 &= -i(sS_1' - S + S_1'),\\ \bar{X}_3 &= -(sS_1' + S' - S_1),\\ \bar{X}_4 &= i(sS_1' - S' - S_1). \end{aligned}$$

If now we use (50′) instead of (49), we get the surface

$$(52)\qquad \begin{aligned} X_1 &= \tfrac{1}{2}[sS' - S - S_1'] + \tfrac{1}{2}[tT' - T - T_1'],\\ X_2 &= -\frac{i}{2}[sS' - S + S_1'] + \frac{i}{2}[tT' - T + T_1'],\\ X_3 &= -\tfrac{1}{2}[sS_1' + S - S_1] - \tfrac{1}{2}[tT_1' + T' - T_1],\\ X_4 &= \frac{i}{2}[sS_1' - S' - S_1] - \frac{i}{2}[tT_1' - T' - T_1], \end{aligned}$$

*and this surface is real* when $s$ and $t$ are chosen conjugate imaginaries, as an easy calculation will show. Hence we have the

THEOREM. *To the imaginary translation-surface in $S_4$ which is the middle surface of an isotropic congruence there corresponds always a real surface obtained by reflecting in one of the coördinate spaces.*

The real surface (52) is not a "minimal" surface, that is, the middle surface of an isotropic congruence. It may be constructed as follows: From the point $S_0$, the projection on the space $X_4 = 0$ of a point on the curve (50′), we draw a parallel $S_0M$ to the chord $ST$ which joins a point $S$ of (50) to a point $T$ on (51). We project $P$, which is a point on the surface (50), drawing the line $PP'$ which meets the line $S_0M$ at $P'$; this point is then a point on the surface (52). (See figure 3.) We notice that $P'$ is the middle point of a chord joining $S'$, the reflection of $S$, to $T$.

Two curves, (49) and (51), being given, there exist two real translation-surfaces whose generating curves are minimal curves. To a point $Q$ on one real

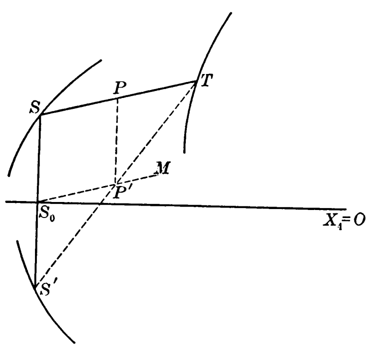


FIG. 3.

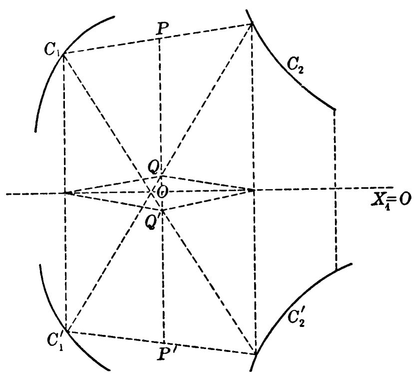


FIG. 4.

surface obtained by using the curves $C_2$ and $C_2'$ corresponds a point $Q'$, the reflection of $Q$, on another real surface obtained by using the curves $C_1$, $C_2$ and $C_1'$ ($C_1'$ and $C_2'$ are reflections of $C_1$ and $C_2$ with respect to the space $X_4 = 0$).

The two real surfaces are therefore reflections of each other. In the same way, the imaginary translation-surfaces determined by $C_1$, $C_2$ and $C_1'$, $C_2'$ are reflections. If the translation-surface is contained in the space $X_4 = 0$, the points $P$, $Q$, $P'$, $Q'$ all coincide at $O$, and the surface becomes a minimal surface in ordinary space (Fig. 4). To these real surfaces belong the "*R*-Flächen" studied by K. Kommerell.* These so-called "minimal" surfaces are,

* K. KOMMERELL, l. c., pp. 543, 596. The equations of the *R*-Flächen are:

$$X_1 = \frac{s+t}{2}, \quad X_2 = \frac{i(s-t)}{2}, \quad X_3 = \tfrac{1}{2}(S+T), \quad X_4 = -\frac{i}{2}(S-T).$$

by our definition, not minimal, since they can only become middle surfaces when contained in the space $X_4 = 0$, in which case they become ordinary minimal surfaces.

## § 7. *Line and sphere geometry.*

We have found (p. 405) that to the $\infty^4$ hyperboloids

$$F = ast + Cs + ct + d$$

in the space $(F, s, t)$ correspond $\infty^4$ minimal cones in $S_4$. If now we transform the space $(F, s, t)$ by means of Euler's transformation

$$(1) \quad x = F'_s, \quad y = t, \quad z = F - sF'_s, \quad p = -s, \quad q = F'_t,$$

the hyperboloids will be transformed into the $\infty^4$ lines of ordinary space, viz.:

$$(2) \qquad x = ay + b, \quad z = cy + d,$$

to which must be added the following:

$$(2') \qquad q = -ap + c,$$

which determines the relation between the coördinates $p$, $q$ and $-1$ of the planes of the $\infty^2$ surface-elements of the line (2), so that to the $\infty^2$ surface-elements of a given hyperboloid there correspond the $\infty^2$ surface-elements of the line. We have thus a one-to-one correspondence between the $\infty^4$ minimal cones and the $\infty^4$ lines of a three-dimensional space $S_3$. To a line $(a, b, c, d)$ in $S_3$ corresponds a minimal cone whose vertex is (p. 404)

$$(3) \quad \alpha = -\frac{b+c}{2}, \quad \beta = \frac{i(c-b)}{2}, \quad \gamma = \frac{d-a}{2}, \quad \delta = \frac{i(a+d)}{2}.$$

To two lines that intersect there correspond two minimal cones that are tangent to each other; in fact, the condition that two lines

$$(4) \qquad \begin{aligned} x &= ay + b, \quad z = cy + d, \\ x &= a'y + b, \quad z = c'y + d, \end{aligned}$$

shall intersect is

$$(5) \qquad (a' - a)(d - d') - (c' - c)(b - b') = 0.$$

The coördinates of the vertices of the corresponding cones are

$$\alpha = -\frac{b+c}{2}, \quad \beta = \frac{i(c-b)}{2}, \quad \gamma = \frac{d-a}{2}, \quad \delta = \frac{i(a+d)}{2},$$

$$\alpha' = -\frac{b'+c'}{2}, \quad \beta' = \frac{i(c'-b')}{2}, \quad \gamma' = \frac{d'-a'}{2}, \quad \delta' = \frac{i(a'+d')}{2},$$

from which we obtain by solving

$$a = -(\gamma + i\delta), \qquad b = -(\alpha - i\beta), \qquad c = -(\alpha + i\beta), \qquad d = \gamma - i\delta,$$

$$a' = -(\gamma' + i\delta'), \qquad b' = -(\alpha' - i\beta'), \qquad c' = -(\alpha' + i\beta'), \qquad d' = \gamma' - i\delta'.$$

Substituting these values in (5), we have

$$(6) \qquad (\alpha' - \alpha)^2 + (\beta' - \beta)^2 + (\gamma' - \gamma)^2 + (\delta' - \delta)^2 = 0,$$

which is the condition that two minimal cones shall be tangent to each other. In particular, if the two lines are consecutive, the relation (6) becomes

$$(7) \qquad d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 = 0.$$

If the line $(a, b, c, d)$ be fixed and the line $(a', b', c', d')$ variable, the locus of the corresponding variable vertex is a minimal cone. Hence, to all the $\infty^3$ lines of a special null-system in $S_3$ correspond the $\infty^3$ minimal cones in $S_4$ whose vertices lie on another minimal cone.

The equations (2) and (2′) show that if the coördinates $y$ and $p$ of the surface-element are fixed, the remaining three, viz.: $x$, $z$ and $q$, are determined. But fixing $y$ and $p$ also fixes $s$ and $t$. Therefore to all the $\infty^2$ surface-elements of a line in $S_3$ correspond the $\infty^2$ minimal lines of a minimal cone in $S_4$.

Let there be given in $S_3$ a non-special null-system which may be written

$$Ad - Bb + C(bc - ad) + Da + Ec + G = 0;$$

to it will correspond in $S_4$ the sphere

$$(8) \quad \alpha^2+\beta^2+\gamma^2+\delta^2+\frac{B-E}{C}\alpha-\frac{i(B+E)}{C}\beta+\frac{A-D}{C}\gamma-\frac{i(A+D)}{C}\delta+\frac{G}{C}=0.$$

which may be written

$$(9) \quad \left(\alpha-\frac{B-E}{2C}\right)^2+\left(\beta-\frac{i(B+E)}{2C}\right)^2+\left(\gamma-\frac{D-A}{2C}\right)^2+\left(\delta-\frac{i(A+D)}{2C}\right)^2 = -\frac{BE+AD+CG}{C^2}.$$

Hence, to all the $\infty^3$ lines of a non-special null-system in $S_4$ correspond the $\infty^3$ minimal cones in $S_4$ having their vertices on a fixed sphere. If we consider each point of the sphere (9) as the carrier of a minimal cone, we have a one-to-one correspondence between the $\infty^5$ null-systems of ordinary space and the $\infty^5$ spheres of four-dimensional space. If the null-system is special,

$$BE + AD + CG = 0,$$

and this means that the fixed sphere (19) is a minimal cone.

The geometry of a linear complex in $S_3$ is thus seen to be equivalent to the geometry of a complex of minimal cones in $S_4$. It should be noted, however, that the null-system must be conceived of as an aggregate of surface-elements, that is to say, the $\infty^3$ lines and their corresponding surface-elements of which each line has $\infty^2$. To each surface-element of a line belonging to the null-system corresponds a minimal line passing through a point on the fixed sphere, that is, to a surface-element corresponds a minimal element. We shall return to this interesting fact later and proceed now to explain briefly the distribution of the linear elements of the minimal cone.

We have shown, p. 405, that $s$ and $t$ may be regarded as curvilinear coördinates on the imaginary sphere at infinity, which we shall denote by $S_\infty$, the curves $s =$ const. and $t =$ const. being the generators of the sphere. By Euler's transformation, $y = t$, $p = -s$; hence all the surface-elements of the line (2), p. 422, for which $y =$ const., give minimal lines through the vertex of the minimal cone in $S_4$ which meet a $t$-generator of $S_\infty$. Similarly for the surface-elements $p =$ const. The minimal lines through the vertex of the minimal cone are thus arranged in two families of planes corresponding to the two families of generators $t$ and $s$ on $S_\infty$. These planes are the minimal planes of the cone.

## § 8. *Lie's line-sphere geometry.*

We shall now return to the congruence of minimal lines (5), § 1, and write it as follows:

$$\begin{aligned} &tX_1 + itX_2 - X_3 + iX_4 + F - sF'_s = 0, \\ (10) \qquad &X_1 - iX_2 + tX_3 + itX_4 + F'_s = 0, \\ &X_1 + iX_2 + sX_3 + isX_4 + F'_t = 0. \end{aligned}$$

Introducing Euler's transformation (1), this congruence becomes:

$$\begin{aligned} &yX_1 + iyX_2 - X_3 + iX_4 + z = 0, \\ (11) \qquad &X_1 - iX_2 + yX_3 + iyX_4 + x = 0, \\ &X_1 + iX_2 - pX_3 - ipX_4 + q = 0, \end{aligned}$$

which equations solved for $X_1$, $X_2$ and $X_3$ may be written

$$(12)\qquad \begin{aligned} X_1 &= \frac{i(y-p)}{1-py}X_4 - \frac{x}{2} - \frac{y(z-yq)+q-pz}{2(1-py)}, \\ X_2 &= \frac{y+p}{1-py}X_4 - \frac{y}{2} - i\frac{y(z+yq)-q+pz}{2(1-py)}, \\ X_3 &= \frac{i(1+py)}{1-py} + \frac{z-py}{1-py}. \end{aligned}$$

The condition that two consecutive minimal lines shall intersect we obtain as before by differentiating the system (11) with respect to the parameters $x$, $y$, $z$, $p$, and $q$. We have

$$(13)\qquad \begin{aligned} (X_1 + iX_2)dy + dz &= 0, \\ (X_3 + iX_4)dy + dx &= 0, \\ -(X_3 + iX_4)dp + dq &= 0. \end{aligned}$$

Eliminating the coördinates $X_i$ from (11) and (13) we find the following conditions:

$$(14)\qquad \begin{aligned} dz - pdx - qdy &= 0, \\ dp\,dx + dq\,dy &= 0. \end{aligned}$$

The first equation expresses the condition that two surface-elements corresponding to the two consecutive minimal lines shall be united. The second expresses the condition that the line of intersection of the planes of the surface-elements joins the two consecutive points $x$, $y$, $z$, and $x + dx$, $y + dy$, $z + dz$. If therefore we have in $S_4$ a minimal curve, there will correspond to it in $S_3$ an element-band the planes of whose surface-elements osculate the point-locus of the band. To a surface $z = f(x, y)$ in $S_3$ corresponds a minimal congruence $\phi(X_1, X_2, X_3, X_4) = 0$. To the two minimal developables determined by a generator $D$ of the congruence correspond two asymptotic lines or bands on the surface $z = f(x, y)$, one of each family; hence, *to the two families of* $\infty^1$ *minimal developables contained in the congruence* $\phi = 0$ *correspond in* $S_3$ *the two families of asymptotic bands on the surface, and to a minimal curve of the developable* (*an edge of regression*) *corresponds an asymptotic curve.*

We shall now project the space $S_4$ into a space $\bar{S}_3$ by means of the transformation (40), p. 415, which transforms the minimal cone

$$(15) \qquad (X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4-\delta)^2=0$$

into a sphere in ordinary space $(X_1, X_2, X_3)$, viz.:

$$(16) \qquad (X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+\delta^2=0$$

$(\alpha, \beta, \gamma)$ being the center and $\pm i\delta$ the radius. We notice that the correspondence is a one-to-two owing to the double sign of the radius $R = \pm i\delta$, that is to say, to the cone corresponds a sphere, while to the sphere correspond two cones, viz.:

$$(17) \qquad (X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4-iR)^2=0$$

and

$$(17') \qquad (X_1-\alpha)^2+(X_2-\beta)^2+(X_3-\gamma)^2+(X_4+iR)^2=0.$$

To a minimal line in $S_4$ corresponds a surface-element in $S_3$, while to a minimal cone $(\alpha, \beta, \gamma, \delta)$ corresponds a sphere of radius $i\delta$ considered as an ensemble of its $\infty^2$ surface-elements. Further, to a minimal congruence will correspond a surface and to the $\infty^1$ minimal developables the $\infty^1$ lines of curvature on the surface, these being considered as bands (curvature bands). Comparing now the two spaces $S_3$ and $\bar{S}_3$ with $S_4$, we see that the same configuration in $S_4$ has been transformed into two different configurations in $S_3$ and $\bar{S}_3$ of such a nature that lines in $S_3$ correspond to spheres in $\bar{S}_3$ and asymptotic lines to lines of curvature. Further, since by the transformation (40), p. 415, the relation (6), § 7, becomes

$$(\alpha-\alpha')^2+(\beta-\beta')^2+(\gamma-\gamma')^2=(R-R')^2,$$

it follows also that to lines that intersect in $S_3$ correspond spheres that touch. The relation between the surface-elements of $S_3$ and $\bar{S}_3$ must therefore be of such a nature that a contact-transformation transforms the elements of $S_3$ into those of $\bar{S}_3$. This contact-transformation is obtained at once by putting $X_4 = 0$ in equations (12), p. 424, viz.:

$$(18) \qquad \begin{aligned} X_1 &= -\frac{x}{2}-\frac{y(z-yq)+q-pz}{2(1-py)}, \\ X_2 &= -\frac{iy}{2}-i\frac{y(z+yq)-q+pz}{2(1-py)}, \\ X_3 &= \frac{z-py}{1-py}. \end{aligned}$$

If to (18) we add the two equations giving the values of the coördinates $P_1$ and $P_2$ (obtained from (12)), we have

$$P_1 = \frac{y-p}{1+yp}, \qquad P_2 = \frac{-i(y+p)}{1+yp}. \tag{19}$$

Equations (18) and (19) give the required contact-transformation, which we recognize as the well known line-sphere contact transformation of Lie. This method of deriving the transformation is, we believe, new and perhaps not without interest.

In conclusion we give the following table showing the relation between the elements of the three spaces $S_3$, $\bar{S}_3$, and $S_4$:

| SPACE $S_3(X_1, X_2, X_3, P_1, P_2)$. | SPACE $S_4(X_1, X_2, X_3, X_4)$. | SPACE $\bar{S}_3(x, y, z, p, q)$. |
|---|---|---|
| I. A surface-element. | I. A minimal line. | I. A surface-element. |
| II. *a*) $dX_3 - P_1 dX_1 - P_2 dX_2 = 0$. *b*) $(dX_1 + P_1 dX_3) dP_1 - (dX_2 + P_2 dX_3) dP_2 = 0$. | II. Intersecting consecutive minimal lines. | II. *a*) $dz - pdx - qdy = 0$. *b*) $dpdx + dqdy = 0$. |
| III. The $\infty^2$ surface-elements of a sphere, center $\alpha$, $\beta$, $\gamma$, radius $\pm i\delta$. | III. The $\infty^2$ minimal lines of a minimal cone, vertex $\alpha$, $\beta$, $\gamma$, $\delta$. | III. The $\infty^2$ surface-elements of the line $x = -(\gamma + i\delta)y - (\alpha - i\beta)$, $y = -(\alpha + i\beta)y + \gamma - i\delta$, $q = (\gamma + i\delta)p - (\alpha + i\beta)$. |
| IV. The $\infty^1$ element bands of a sphere whose point-loci are the rectilinear generators of the sphere. | IV. The $\infty^1$ plane pencils of a minimal cone; two families, $p=$const., $q=$const. | IV. The $\infty^1$ surface-elements of a point, or a "hinge," and the $\infty^1$ surface-elements of a line (flat band). |
| V. A spherical band. | V. A two-dimensional minimal cone, vertex $\alpha$, $\beta$, $\gamma$, $\delta$. | V. A twisted band. |
| VI. A surface. | VI. A minimal or isotropic congruence. | VI. A surface. |
| VII. An ensemble of $\infty^1$ surface-elements forming a curvature band on the surface VI. | VII. A minimal developable. | VII. An ensemble of $\infty^1$ surface-elements forming an asymptotic band on the surface VI. |
| VIII. The double family of curvature bands on VI. | VIII. The $\infty^1$ minimal developables contained in an isotropic congruence VI. (A double family, since each generator determines two developables.) | VIII. The double family of asymptotic lines on VI. |

IX. A congruence of $\infty^2$ spheres, the so-called curvature spheres, belonging to the surface VI.

X. System of $\infty^3$ spheres cutting a fixed sphere at a constant angle. (See LIE-SCHEFFERS, loc. cit., vol. I, pp. 651–654.)

IX. The minimal congruence considered as an aggregate of $\infty^2$ minimal cones, each cone having a line-element in common with a line of the congruence on an edge of regression.

X. System of $\infty^3$ minimal cones having their vertices on a fixed sphere.

IX. A congruence of $\infty^2$ lines consisting of the $\infty^2$ linear tangents along the asymptotic lines.

X. Linear complex, or null-system.

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